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Analysis of Oscillation in Nonlinear Neutral Delay Differential Equations

Mrs.N.Prashanthi
Assistant Professor, Department of H&S,
Malla Reddy College of Engineering for Women., Maisammaguda., Medchal., TS, India

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Introduction

We will study the first order nonlinear neutral delay differential equation in this work.

$$(x(t) - q(t)x(t-\sigma))' + f(t, x(\tau(t))) = 0, \quad t \geq t_0, \tag{1}$$

where

$$q, \tau \in C([t_0, \infty), \mathbf{R}^+), \quad \sigma \in (0, \infty) \quad \tau(\mathrm{t}) < \mathrm{t}, \quad \lim_{t \to \infty} \tau(\mathrm{t}) = \infty, \tag{2}$$

and

$$\sum_{i=1}^{n} \prod_{j=1}^{i} \frac{1}{q(t_j)} \longrightarrow \infty \quad as \quad n \longrightarrow \infty, \tag{3}$$

$$f \in C([t_0, \infty) \times \mathbf{R}, \mathbf{R}), \quad \mathrm{uf(t,u)} \geq 0, \quad \mathrm{q(t)} \neq 1. \tag{4}$$

In connection with the nonlinear function $f(t,u)$ in (1) we suppose that the following assumption (H) holds :

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(H) There are a piecewise continuous functions

$$p : [t_0, \infty) \to \mathbf{R}^+ = [0, \infty), \ \mathrm{g} \in \mathrm{C}(\mathbf{R}, \mathbf{R}^+)$$

and a number $\varepsilon_0 > 0$ such that

(i) g is nondecreasing on $\mathbf{R}^+$

(ii) $g(-u) = g(u)$, $\lim_{u \to 0} g(u) = 0$,

(iii) $\int_0^\infty g(e^{-u})\, du < \infty$,

(iv) $\frac{1}{|u|} |f(t,u) - p(t)u| \leq p(t) g(u)$ for $t \geq t_0$ and $0 < |u| < \varepsilon_0$,

(v) For each $\varphi \in C([t_0, \infty), \mathbf{R})$ with $\lim_{t \to \infty} \varphi(t) > 0$,

$$\int_{t_0}^{\infty} f(t, \varphi(\tau(t)))\, dt = \infty, \ \int_{t_0}^{\infty} f(t, -\varphi(\tau(t)))\, dt = -\infty.$$

A solution $x(t)$ of equation (1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_0, \infty)$. Otherwise it is nonoscillatory and the equation (1) is called oscillatory if every solution of this equation is oscillatory.

When $q(t) = 0$, Eq.(1) reduces to the equation

$$x'(t) + f(t, x(\tau(t))) = 0,$$

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which has been studied recently by Tang and Shen [32]. The oscillatoi havior of various other neutral delay differential equations have been ii gated by many authors. For contributions we refer the reader to the p $[1-3,\ 6-12,\ 14-27,\ 29-36]$, and references cited therein.
Also, in a recent papers Elabbasy and Saker [6], Kubiaczyk and Saker [1' their references obtained an infinite integral conditions for oscillation of the neutral delay differential equation

$$(x(t)-q(t)x(t-\sigma))'+p(t)x(t-\tau)=0,$$

Our aim in this paper is to extend the results in [7, 18] and provide som finite and infinite integral sufficient conditions for the oscillation of all sol of (1). Some examples that are illustrating our main results are given.

The following notations will be used throughout this paper,

$\delta(t)=\max\{\tau(t):t_0\le s\le t\}$ and $\delta^{-1}(t)=\min\{s\ge t_0:\delta(s)=t\}$.

Clearly, δ and δ^{-1} are nondecreasing and satisfy

(A) $\delta(t)<t$ and $\delta^{-1}(t)>t$,

(B) $\delta\left(\delta^{-1}(t)\right)=t$ and $\delta^{-1}(\delta(t))\le t$.

Let $\delta^{-k}(t)$ be defined on $[t_0,\infty)$ by

$$\delta^{-(k+1)}(t)=\delta^{-1}\left(\delta^{-k}(t)\right),\quad k=1,2,...$$

Also, we use the sequence $\{p_k\}$ of functions defined as follows:

$$p_1(t)=\int_t^{\delta^{-1}(t)}p(s)\,ds,\quad t\ge t_0,$$

$$p_{k+1}(t)=\int_t^{\delta^{-1}(t)}p(s)\,p_k(s)\,ds,\quad t\ge t_0,\ k=1,2,...$$

In what follows, when we write a functional inequality we will mean holds for all sufficiently large values of t.

2. Main results

To prove our main results we shall need the following Lemmas.

Lemma 1. *Assume that* (2), (3) *and* (4) *hold, let* $x(t)$ *be an eventually*] *solution of* (1) *and set*

$$z(t)=x(t)-q(t)x(t-\sigma).$$

Then $z(t)$ *is eventually nonincreasing positive function.*

Proof. From (1), (4), we have $z'(t)=-f(t,x(\tau(t)))\le 0$ eventually. W that $z(t)$ is a positive function. If not, then there exist $T\ge t_0$ and $\alpha<$ that $z(t)<\alpha$ for $t\ge T$. Then from (6), we have

$$x(t)<\alpha+q(t)x(t-\sigma),$$

which implies that

$$x(t+\sigma)<\alpha+q(t+\sigma)x(t).$$

Now we choose k such that $t_k=t^*+k\sigma>T$. Then $x(t_{k+1})<\alpha+q(t_{k+1})$

Applying this inequality by induction it gives

$$x(t_n)<\alpha\left[1+\sum_{i=k+2}^{n}\prod_{j=0}^{n-i}q(t_{n-j})\right]+\prod_{i=k+1}^{n}q(t_i)x(t_k).$$

Now define q_n and d_n by

$$q_n=1+\sum_{i=k+2}^{n}\prod_{j=0}^{n-i}q(t_{n-j}),\quad d_n=\prod_{i=k+1}^{n}q(t_i),$$

and let

$$s_n=\sum_{i=1}^{n}\prod_{j=1}^{i}\frac{1}{q(t_j)}.$$

Then

$$s_n^*=\frac{q_n}{d_n}=\left(s_n-\sum_{i=1}^{k+1}\prod_{j=1}^{i}\frac{1}{q(t_j)}\right)q(t_{k+1})...q(t_1)\to\infty\quad\text{as } n\to\infty,$$

by condition (3). Using the last inequality, one can see that

$$x(t_n)<\left[s_n^*+\frac{x(t_k)}{\alpha}\right]\alpha d_n\to-\infty\quad\text{as } n\to\infty,$$

and this contradicts the assumption that $x(t)>0$. Then $z(t)$ must be positive function. The proof is complete. □

Note that the proof of Lemma 1 is similar to that of Lemma 1 in [4] and we state it here for the sake of completeness.

Lemma 2. *Assume that* (2), (3), (4) *and* (H) *hold, let* $x(t)$ *be an eventually positive solution of* 1. *Then* $x(t)$ *and* $z(t)$ *are convergent to zero monotonically as* $t\to\infty$.

Proof. By Lemma 1 $z(t)$ is a nonincreasing positive function and satisfy the equation

$$z'(t)=-f(t,x(\tau(t))).$$

Choose a $t_1\ge t_0$ such that $x(t)>0$, $z(t)>0$ for $t\ge t_1$.

It follows from equations (2), (4) and (H) that there exists $t_2>t_1$ such that $\tau(t)\ge t_1$ and $z'(t)\le 0$ for $t>t_2$.

Hence

$$\lim_{t\to\infty}x(t)\ge\lim_{t\to\infty}z(t)=\alpha\ge 0\text{ exists.}$$

If $\alpha>0$, then from (1), we have

$$z(t)-z(t_0)=-\int_{t_0}^{t}f(t,x(\tau(s)))\,ds.$$

It follows from the assumption (H)(v) that $\lim_{t\to\infty}z(t)=-\infty$, which contradicts that $z(t)$ being positive. Then $\lim_{t\to\infty}z(t)=0$. Since $q(t)\neq 1$, we have also $\lim_{t\to\infty}x(t)=0$. Then the proof is complete. □

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Lemma 3. *Assume that* (2), (3), (4) *and* (H) *hold. If* $x(t)$ *is a nonoscillatory solution of equation* (1), *there exist* $A > 0$, $\varepsilon > 0$ *and* $T \in (0, \infty)$ *such that for* $t \geq T$

$$|x(t)| \leq A \exp\left(-\frac{1}{2}\int_T^t p(s)\,ds\right) + \varepsilon. \qquad (7)$$

Proof. We shall assume $x(t)$ to be eventually positive [if $x(t)$ is eventually negative the proof is similar]. By Lemma 2, there exists $t_1 > 0$ such that

$$0 < x\big(\tau(t)\big) < \varepsilon_0 \quad \text{for } t \geq t_1.$$

From (H), we find that for $t \geq t_1$

$$f(t, x(\tau(t))) \geq p(t)\left[1 - g(x(\tau(t)))\right] x(\tau(t)),$$

and $\lim_{t\to\infty} x(t) = 0$. By assumption (H), there exists $T > t_1$ such that for $t \geq T$

$$f(t, x(\tau(t))) \geq \frac{1}{2}p(t)\,x(\tau(t)) \geq \frac{1}{2}p(t)\,x(t),$$

and it follows from (1) that for $t \geq T$

$$(x(t) - q(t)x(t-\sigma))' + \frac{1}{2}p(t)\,x(t) \leq 0,$$

and

$$z'(t) + \frac{1}{2}p(t)\,z(t) \leq 0,$$

where $z(t) = x(t) - q(t)x(t-\sigma)$. This yields for $t \geq T$

$$z(t) \leq A \exp\left[-\frac{1}{2}\int_T^t p(s)\,ds\right],$$

and

$$|x(t)| \leq A \exp\left(-\frac{1}{2}\int_T^t p(s)\,ds\right) + \varepsilon,$$

where $A = x(T) - q(T)x(T-\sigma)$. The proof is complete.

Lemma 4. *Assume that* (2), (3), (4) *and* (H) *hold. If equation* (1) *nonoscillatory solution, then eventually*

$$\int_{\tau(t)}^t p(s)\,ds \leq 2 \quad \textit{and} \quad p_k(t) \leq 2^k, \quad k = 1, 2, \ldots$$

Proof. Let us suppose that $x(t)$ is a nonoscillatory solution of equation (1) which we shall assume to be eventually positive [if $x(t)$ is eventually negative the proof is similar]. By Lemma 2, there exists $T \geq 0$ such that

$$x(\tau(t)) \geq x(t) > 0 \quad \text{for } t \geq T,$$

$$(x(t) - q(t)x(t-r))' + \frac{1}{2}p(t)\,x(\tau(t)) \leq 0,$$

and

$$z'(t) + \frac{1}{2}p(t)\,z(\tau(t)) \leq 0 \quad \text{for } t \geq T. \qquad (9)$$

Integrating both sides from $\tau(t)$ to t yields that

$$z(t) - z(\tau(t)) + \frac{1}{2}\int_{\tau(t)}^t p(s)\,z(\tau(s))\,ds \leq 0 \quad \text{for } t \geq T.$$

By the decreasing nature of $z(t)$ for large t and the increasing nature of $\tau(t)$, there exists $T_1 \geq T$ such that

$$z(t) - z(\tau(t)) + \frac{1}{2}z(\tau(t))\int_{\tau(t)}^t p(s)\,ds \leq 0 \quad \text{for } t \geq T_1.$$

Then, we have

$$\int_{\tau(t)}^t p(s)\,ds \leq 2.$$

Also, Integrating both sides of equation (9) from t to $\delta^{-1}(t)$ yields

$$z\left(\delta^{-1}(t)\right) - z(t) + \frac{1}{2}\int_t^{\delta^{-1}(t)} p(s)\,z(\tau(s))\,ds \leq 0 \quad \text{for } t \geq T.$$

By the decreasing nature of $z(t)$ for large t and the increasing nature of $\tau(t)$, there exists $T_1 \geq T$ such that

$$z\left(\delta^{-1}(t)\right) - z(t) + \frac{1}{2}\left(\int_t^{\delta^{-1}(t)} p(s)\,ds\right) z\left(\tau\left(\delta^{-1}(t)\right)\right) \leq 0 \quad \text{for } t \geq T_1.$$

or

$$z\left(\delta^{-1}(t)\right) - z(t) + \frac{1}{2}\left(\int_t^{\delta^{-1}(t)} p(s)\,ds\right) z(t) \leq 0 \quad \text{for } t \geq T_1.$$

Then, we have

$$p_1(t) = \int_t^{\delta^{-1}(t)} p(s)\,ds \le 2.$$

By iteration we deduce from this that

$$p_k(t) \le 2^k,$$

which shows that (8) holds for $t \ge T_1$. The proof of Lemma 4 is complete. □

Lemma 5. *Assume that* (2), (3), (4) *and* (*H*) *hold, and that*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 0. \tag{10}$$

If $x(t)$ is a nonoscillatory solution of equation (1), *then* $\frac{z(\tau(t))}{z(t)}$, *which is well defined for large t, is bounded.*

Proof. Let us suppose that $x(t)$ is a nonoscillatory solution of equation (1) which we shall assume to be eventually positive [if $x(t)$ is eventually negative the proof is similar]. By the same argument as in the proof of Lemma 3, there exists $T > 0$, such that

$$x(\tau(t)) \ge x(t) > 0 \quad \text{for } t \ge T,$$

$$(x(t) - q(t)x(t-\sigma))' + \frac{1}{2}p(t)x(\tau(t)) \le 0,$$

and

$$z'(t) + \frac{1}{2}p(t)z(\tau(t)) \le 0 \quad \text{for } t \ge T.$$

The rest of the proof is similar to that of Lemma 5 in [24] respectively, and hence is omitted. □

Theorem 1. *In addition to the assumptions* (2), (3), (4) *and* (*H*) *assume that*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} P(s)\,ds > \frac{1}{e},$$

or

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} P(s)\,ds > 1,$$

where $P(t) = (1-\epsilon)p(t)$. Then every solution of Eq. (1) *oscillates.*

Proof. Let us suppose that $x(t)$ is a nonoscillatory solution of equation (1) which we shall assume to be eventually positive [if $x(t)$ is eventually negative the proof is similar]. By Lemma 2, there exists $t_1 > 0$ such that

$$0 < x(\tau(t)) < \varepsilon_0 \quad \text{for } t \ge t_1.$$

From (H), we find that for $t \ge t_1$

$$f(t, x(\tau(t))) \ge p(t)[1 - g(x(\tau(t)))]x(\tau(t)),$$

and $\lim_{t\to\infty} x(t) = 0$. By assumption (H), there exists $T > t_1$ such that for $t \ge T$

$$f(t, x(\tau(t))) \ge (1-\epsilon)p(t)x(\tau(t)),$$

and it follows from (1) that for $t \ge T$

$$(x(t) - q(t)x(t-\sigma))' + (1-\epsilon)p(t)x(\tau(t)) \le 0,$$

and

$$z'(t) + P(t)z(\tau(t)) \le 0. \tag{13}$$

But, then by Corollary 3.2.2 [11] the delay differential equation

$$z'(t) + P(t)z(\tau(t)) = 0, \tag{14}$$

has an eventually positive solution as well. It is also well known that (11) or (12) implies (14) has no eventually positive solution (see, [11] Theorem 3.4.3). This contradiction completes the proof. □

Remark 1. It is clear that every solution of (1) oscillates if (14) has no eventually positive solutions.

It is clear that there is a gap between (11) and (12) for the oscillation of all solutions of (1). The problem how to fill this gap for the equation (1) when the limit

$$\lim_{t\to\infty} \int_{\tau(t)}^{t} P(s)\,ds,$$

does not exist needs to be considered. This problem has been cleared for the linear Eq. (14). Let the numbers k and l be defined by

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} P(s)ds, \; l = \limsup_{t\to\infty} \int_{\tau(t)}^{t} P(s)ds, \; .$$

$$0 < k \le \frac{1}{e}, \; l < 1,$$

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and λ is the smallest root of the equation $\lambda = e^{k\lambda}$. Then Eq. (14) will be oscillatory if either of the following conditions is satisfied:

$$(C_1) \quad l > \frac{\ln\lambda + 1}{\lambda}, \qquad [18]$$

$$(C_2) \quad l > 1 - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2}, \qquad [34]$$

$$(C_3) \quad l > \frac{1 + \ln\lambda}{\lambda} - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2}, \qquad [14]$$

$$(C_4) \quad l > 2k + \frac{2}{\lambda} - 1, \qquad [19]$$

$$(C_5) \quad l > \frac{\ln\lambda - 1 + \sqrt{5 - 2\lambda + 2k\lambda}}{\lambda}, \qquad [30]$$

$$(C_6) \quad l > \frac{e-1}{e-2}\left(k + \frac{1}{\lambda_1}\right) - \frac{1}{e-2}, \qquad [7]$$

Remark 2. Theorem 1 implies that Eq. (1) will also be oscillatory if either of the conditions $(C_1) - (C_6)$ is satisfied.

In the following theorems we present new infinite integral sufficient conditions for the oscillation of all solutions of (1)

Theorem 2. *Assume that* (2), (3), (4), (10) *and* (H) *hold, and suppose that there exists a positive integer* n *such that*

$$\int_{t_0}^{\infty} p(t)\ln(p_n(t) + 1)\,dt = \infty. \qquad (15)$$

Then every solution of (1) *oscillates.*

Proof. Assume that (1) has a nonoscillatory solution $x(t)$ which will be assumed to be eventually positive (if $x(t)$ is eventually negative the proof is similar). By Lemma 2 and assumption (H), there exists $t_0^* \geq t_0$ such that

$$0 < x(t) \leq x(\delta(t)) \leq x(\tau(t)) < \varepsilon_0, \quad g(x(\tau(t))) < 1, \quad t \geq t_0^*,$$

where ε_0 is given by assumption (H). (16) and (H) yield that for $t \geq t_0^*$,

$$f(t, x(\tau(t))) \geq p(t)[1 - g(x(\tau(t)))]\,x(\tau(t)) \geq p(t)[1 - g(x(\tau(t)))]\,z(\delta($$

and it follows from (1) that

$$\frac{z'(t)}{z(t)} + p(t)\frac{z(\delta(t))}{z(t)}[1 - g(x(\tau(t)))] \leq 0, \quad t \geq t_0^*.$$

By Lemmas 1 – 5, there exist $T > t_2$, $A > 0$, $\varepsilon > 0$ and $M > 0$ such that for $t \geq T$,

$$x(\tau(t)) \leq A\exp\left(-\frac{1}{2}\int_T^{\tau(t)} p(s)\,ds\right) + \varepsilon, \qquad (19)$$

$$\int_{\delta(t)}^{t} p(s)\,ds \leq \int_{\tau(t)}^{t} p(s)\,ds \leq 2, \quad p_k(t) \leq 2^k, \quad k = 1, 2, \ldots, \qquad (20)$$

$$\frac{z(\delta(t))}{z(t)} \leq \frac{z(\tau(t))}{z(t)} \leq M. \qquad (21)$$

Let $t_k = \delta^{-k}(T)$, $k = 1, 2, \ldots$Clearly $t_k \to \infty$ as $k \to \infty$. Set

$$\lambda(t) = \frac{-z'(t)}{z(t)}, \quad t \geq T.$$

Then

$$\frac{z(\delta(t))}{z(t)} = \exp\int_{\delta(t)}^{t} \lambda(s)\,ds, \quad t \geq t_1,$$

and from (18) we have for $t \geq t_1$

$$\lambda(t) \geq p(t)\exp\int_{\delta(t)}^{t} \lambda(s)\,ds - p(t)\,g(x(\tau(t)))\,\frac{z(\delta(t))}{z(t)}. \qquad (22)$$

It follows from (19) – (22) that for $t \geq t_1$,

$$\lambda(t) \geq p(t)\exp\int_{\delta(t)}^{t} \lambda(s)\,ds - Mp(t)\,g\left(A\exp\left(-\frac{1}{2}\int_T^{\tau(t)} p(s)\,ds\right) + \varepsilon\right)$$

$$\geq p(t)\exp\int_{\delta(t)}^{t} \lambda(s)\,ds - Mp(t)\,g\left(A_1\exp\left(-\frac{1}{2}\int_T^{t} p(s)\,ds\right) + \varepsilon\right), \qquad (23)$$

where $A_1 = eA$. By the inequality $e^c \geq c$ for $c \geq 0$, we have for $t \geq t_1$

$$\lambda(t) \geq p(t)\int_{\delta(t)}^{t} \lambda(s)\,ds - Mp(t)\,g\left(A_1\exp\left(-\frac{1}{2}\int_T^{t} p(s)\,ds\right) + \varepsilon\right). \qquad (24)$$

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Set

$$\alpha(t) = \frac{1}{2}\int_T^t p(s)\,ds, \quad t \geq T,$$

and

$$\begin{cases} \lambda_0(t) = \lambda(t), & t \geq T, \\ \lambda_k(t) = p(t)\displaystyle\int_{\delta(t)}^t \lambda_{k-1}(s)\,ds, & t \geq t_k,\ k = 1,2,...,n, \end{cases}$$

and

$$\begin{cases} G_0(t) = 0, & t \geq T, \\ G_k(t) = p(t)\displaystyle\int_{\delta(t)}^t G_{k-1}(s)\,ds \\ \quad + Mp(t)\,g(A_1\exp(-\alpha(t)) + \varepsilon), \ t \geq t_k, \ k = 1,2,...,n. \end{cases}$$

Clearly (10) implies that $\alpha(t)$ is nondecreasing on $[T,\infty)$ and $\alpha(t)$ $t \to \infty$. By iteration we deduce from (24) that

$$\lambda(t) \geq \lambda_k(t) - G_k(t), \quad t \geq t_k,\ k = 1,2,...n-1,$$

and so by (23),

$$\lambda(t) \geq p(t)\exp\left(\int_{\delta(t)}^t \lambda_{n-1}(s)\,ds\right)\exp\left(-\int_{\delta(t)}^t G_{n-1}(s)\,ds\right) - G_1(t),$$

From (27), one can easily obtain

$$G_{k+1}(t) - G_k(t)$$
$$= p(t)\int_{\delta(t)}^t [G_k(s) - G_{k-1}(s)]\,ds, \quad t \geq t_{k+1},\ k = 1,2,...,n-1.$$

By (20), (25) and (27), for $t \geq t_2$ we have

$$\int_{\delta(t)}^t G_1(s)\,ds = M\int_{\delta(t)}^t p(s)\,g(A_1\exp(-\alpha(s)) + \varepsilon)\,ds$$

$$= 2M\int_{\alpha(\delta(t))}^{\alpha(t)} g\left(A_1e^{-u} + \varepsilon\right)du$$

$$\leq 2M\int_{\alpha(t)-1}^{\alpha(t)} g\left(A_1e^{-u} + \varepsilon\right)du.$$

Thus, from (30), we get

$$G_2(t) - G_1(t) = p(t)\int_{\delta(t)}^t G_1(s)\,ds$$

$$\leq 2Mp(t)\int_{\alpha(t)-1}^{\alpha(t)} g\left(A_1e^{-u} + \varepsilon\right)du, \quad t \geq t_2,$$

$$G_3(t) - G_2(t) = p(t)\int_{\delta(t)}^t [G_2(s) - G_1(s)]\,ds$$

$$\leq 2Mp(t)\int_{\delta(t)}^t p(s)\int_{\alpha(s)-1}^{\alpha(s)} g\left(A_1e^{-u} + \varepsilon\right)du\,ds$$

$$= 4Mp(t)\int_{\alpha(\delta(t))}^{\alpha(t)}\int_{v-1}^{v} g\left(A_1e^{-u} + \varepsilon\right)du\,dv$$

$$\leq 4Mp(t)\int_{\alpha(t)-1}^{\alpha(t)}\int_{v-1}^{v} g\left(A_1e^{-u} + \varepsilon\right)du\,dv$$

$$\leq 4Mp(t)\int_{\alpha(t)-2}^{\alpha(t)} g\left(A_1e^{-u} + \varepsilon\right)du, \quad t \geq t_3.$$

By induction, one can prove in general that for $k = 2,3,...,n-1$,

$$G_k(t) - G_{k-1}(t) \leq (2)^{k-1}(k-2)!Mp(t)\int_{\alpha(t)-(k-1)}^{\alpha(t)} g\left(A_1e^{-u} + \varepsilon\right)du, \quad t \geq t_k,$$

and so

$$G_{n-1}(t) = \sum_{k=1}^{n-1}[G_k(t) - G_{k-1}(t)]$$

$$\leq G_1(t) + Mp(t)\sum_{k=2}^{n-1}(2e)^{k-1}(k-2)! \qquad (32)$$

$$\times \int_{\alpha(t)-(k-1)}^{\alpha(t)} g\left(A_1 e^{-u}+\varepsilon\right) du, \quad t \geq t_{n-1}.$$

By (20), (21) and (26), we obtain

$$\begin{cases} \lambda_1(t) = p(t)\int_{\delta(t)}^{t} \lambda(s)\,ds = p(t)\ln\left[\frac{z(\delta(t))}{z(t)}\right] \\ \quad \leq p(t)\ln M, \quad t \geq t_1, \\ \lambda_2(t) = p(t)\int_{\delta(t)}^{t} \lambda_1(s)\,ds \leq p(t)\ln M \int_{\delta(t)}^{t} p(s)\,ds \\ \quad \leq 2p(t)\ln M, \quad t \geq t_2, \\ \ldots\ldots \\ \lambda_{n-1}(t) \leq 2^{n-2}p(t)\ln M, \quad t \geq t_{n-1}. \end{cases}$$

Set

$$D(t) = p(t)\exp\left(\int_{\delta(t)}^{t} \lambda_{n-1}(s)\,ds\right)\left[1-\exp\left(-\int_{\delta(t)}^{t} G_{n-1}(s\right.\right.$$

$$+G_1(t), \quad t \geq t_n.$$

One can easily see that

$$0 \leq 1-e^{-c} \leq c, \quad c \geq 0.$$

From, (20), (32), (33), (33) and (34) we have

$$D(t) \leq p(t)\exp\left(\int_{\delta(t)}^{t} \lambda_{n-1}(s)\,ds\right)\int_{\delta(t)}^{t} G_{n-1}(s)\,ds + G_1(t)$$

$$\leq G_1(t) + p(t)\exp\left(2^{n-2}\ln M \int_{\delta(t)}^{t} p(s)\,ds\right)$$

$$\times \int_{\delta(t)}^{t}\left[G_1(s) + Mp(s)\sum_{k=2}^{n-1}(2)^{k-1}(k-2)! \int_{\alpha(s)-(k-1)}^{\alpha(s)} g\left(A_1e^{-u}+\varepsilon\right)du\right]ds$$

$$\leq G_1(t) + 2Mp(t)\exp\left((2)^{n-1}\ln M\right)\int_{\alpha(t)-1}^{\alpha(t)} g\left(A_1e^{-u}+\varepsilon\right)du$$

$$+Mp(t)\exp\left((2)^{n-1}\ln M\right)$$

$$\times \sum_{k=2}^{n-1}(2)^{k-1}(k-2)!\int_{\delta(t)}^{t} p(s)\int_{\alpha(s)-(k-1)}^{\alpha(s)} g\left(A_1e^{-u}+\varepsilon\right)duds \tag{35}$$

$$\leq G_1(t) + M_1p(t)\sum_{k=1}^{n-1}(2)^{k-1}(k-1)!\int_{\alpha(t)-k}^{\alpha(t)} g\left(A_1e^{-u}+\varepsilon\right)du, \quad t \geq t_n,$$

where $M_1 = 2M\exp\left((2)^{n-1}\ln M\right)$. Let $T^* > t_n$ be such that $\alpha(T^*) > n + \ln A_1$. It follows from (36) and (H) that

$$\int_{T^*}^{\infty} D(t)\,dt \leq \int_{T^*}^{\infty} G_1(t)\,dt + M_1\sum_{k=1}^{n-1}(2)^{k-1}(k-1)!\int_{T^*}^{\infty} p(t)\int_{\alpha(t)-k}^{\alpha(t)} g\left(A_1e^{-u}\right)dudt$$

$$\leq 2M\int_{\alpha(T^*)}^{\infty} g\left(A_1e^{-u}\right)du$$

$$+2M_1\sum_{k=1}^{n-1}(2)^{k-1}(k-1)!\int_{\alpha(T^*)}^{\infty}\int_{v-k}^{v} g\left(A_1e^{-u}\right)dudv$$

$$\leq 2M\int_{\alpha(T^*)}^{\infty} g\left(A_1e^{-u}\right)du + 2M_1\sum_{k=1}^{n-1}(2)^{k-1}k!\int_{\alpha(T^*)-(k+1)}^{\infty} g\left(A_1e^{-u}\right)du$$

$$\int_{T^*}^{\infty} D(t)\,dt \leq 2M\int_{0}^{\infty} g\left(e^{-u}\right)du + 2M_1\sum_{k=1}^{n-1}(2)^{k-1}k!\int_{0}^{\infty} g\left(e^{-u}\right)du < \infty. \tag{36}$$

Since

$$p(t)\exp\left(\int_{\delta(t)}^{t} \lambda_{n-1}(s)\,ds\right)\exp\left(-\int_{\delta(t)}^{t} G_{n-1}(s)\,ds\right) - G_1(t)$$

$$= p(t)\exp\left(\int_{\delta(t)}^{t} \lambda_{n-1}(s)\,ds\right) - D(t), \quad t \geq t_n,$$

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it follows from (29) that

$$\lambda(t) \geq p(t) \exp\left(\int_{\delta(t)}^{t} \lambda_{n-1}(s)\, ds \right) - D(t), \quad t \geq t_n,$$

or

$$\lambda(t) \geq p(t) \exp\left(\frac{1}{p_n(t)} p_n(t) \int_{\delta(t)}^{t} \lambda_{n-1}(s)\, ds \right) - D(t), \quad t \geq t_n.$$

One can easily show that $e^{\gamma x} \geq x + \dfrac{\ln(\gamma+1)}{\gamma}$ for all $x \geq 0$ and $\gamma > 0$, for $t \geq t_n$,

$$p_n(t)\lambda(t) - p(t) \int_{\delta(t)}^{t} \lambda_{n-1}(s)\, ds \geq p(t) \ln(p_n(t)+1) - p_n(t) D(t)$$

For $N > \delta^{-n}(T^*)$, we have

$$\int_{T^*}^{N} p_n(t)\lambda(t)\, dt - \int_{T^*}^{N} p(t) \int_{\delta(t)}^{t} \lambda_{n-1}(s)\, ds dt$$
$$\geq \int_{T^*}^{N} p(t) \ln(p_n(t)+1)\, dt - \int_{T^*}^{N} p_n(t) D(t)\, dt.$$

Let

$$\delta^1(t) = \delta(t), \quad \delta^{k+1}(t) = \delta\left(\delta^k(t)\right), \quad k = 1, 2, ..., n.$$

Then by interchanging the order of integration, we have

$$\int_{T^*}^{N} p(t) \int_{\delta(t)}^{t} \lambda_{n-1}(s)\, ds dt \geq \int_{T^*}^{\delta(N)} \lambda_{n-1}(t) \int_{t}^{\delta^{-1}(t)} p(s)\, ds dt$$
$$= \int_{T^*}^{\delta(N)} p(t) p_1(t) \int_{\delta(t)}^{t} \lambda_{n-2}(s)\, d$$
$$\geq \int_{T^*}^{\delta^2(N)} \lambda_{n-2}(t) \int_{t}^{\delta^{-1}(t)} p(s) p_1($$
$$= \int_{T^*}^{\delta^2(N)} p(t) p_2(t) \int_{\delta(t)}^{t} \lambda_{n-3}(s)\, ds dt$$
$$\cdots\cdots\cdots$$
$$\geq \int_{T^*}^{\delta^n(N)} \lambda(t) p_n(t)\, dt.$$

From this and (39) we have

$$\int_{\delta^n(N)}^{N} p_n(t)\lambda(t)\, dt \geq \int_{T^*}^{N} p(t) \ln(p_n(t)+1)\, dt - \int_{T^*}^{N} p_n(t) D(t)\, dt, \qquad (40)$$

which together with (20) yields

$$2^n \int_{\delta^n(N)}^{N} \lambda(t)\, dt \geq \int_{T^*}^{N} p(t) \ln(p_n(t)+1)\, dt - 2^n \int_{T^*}^{N} D(t)\, dt,$$

or

$$\ln \frac{x(\delta^n(N))}{x(N)} \geq 2^{-n} \int_{T^*}^{N} p(t) \ln(p_n(t)+1)\, dt - \int_{T^*}^{N} D(t)\, dt. \qquad (41)$$

In view of (15) and (36), we have

$$\lim_{N\to\infty} \frac{x(\delta^n(N))}{x(N)} = \infty. \qquad (42)$$

On the other hand, (21) implies that

$$\frac{x(\delta^n(N))}{x(N)} = \frac{x(\delta^1(N))}{x(N)} \cdot \frac{x(\delta^2(N))}{x(\delta^1(N))} \cdots \frac{x(\delta^n(N))}{x(\delta^{n-1}(N))} \leq M^n.$$

This contradicts (42) and completes the proof.

Remark 3. From Lemma 3 we have

$$\liminf_{t\to\infty} p_k(t) \leq (2)^{k-1} \liminf_{t\to\infty} \int_{t}^{\delta^{-1}(t)} p(s)\, ds \leq (2)^{k-1} \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\, ds.$$

As a result, by Theorem 2 we have

Corollary 1. *Assume that (2), (3), (4), and (H) hold, and that there exists a positive integer n such that*

$$\liminf_{t\to\infty} p_n(t) > 0.$$

Then every solution of (1) oscillates.

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